# Folding sequences

M J Dunwoody

**Abstract**   Bestvina and Feighn showed that a morphism $S \to T$ between two simplicial trees that commutes with the action of a group $G$ can be written as a product of elementary folding operations. Here a more general morphism between simplicial trees is considered, which allow different groups to act on $S$ and $T$. It is shown that these morphisms can again be written as a product of elementary operations: the Bestvina–Feighn folds plus the so-called "vertex morphisms". Applications of this theory are presented. Limits of infinite folding sequences are considered. One application is that a finitely generated inaccessible group must contain an infinite torsion subgroup.

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*Dedicated to David Epstein on the occasion of his 60th birthday*

## 1   Introduction

A morphism $\phi\colon S \to T$ of finite trees can be written as a product of elementary *folds*, in which two edges with a common vertex are folded together, and an isomorphism. Bestvina and Feighn [1] have given a generalization of this result. The case they consider is when $S$ and $T$ are (generally infinite) simplicial $G$–trees for which $G\backslash S$ and $G\backslash T$ are finite graphs $T$ is minimal, and $G$ and the edge stabilizers of $T$ in $G$ are finitely generated. The morphism now becomes a product of equivariant folds and an isomorphism. In each such fold a whole orbit of pairs of edges are folded together. Such an operation is easy to describe in terms of its effect on the quotient graph $G\backslash S$ and the edge and vertex stabilizers of $S$. These are specified in a *graph of groups* determined by a labelling of the edges and vertices of $G\backslash S$. In this paper a further generalization is given. We now allow different groups to act on $S$ and $T$. Thus $S$ is a $G$–tree and $T$ is an $H$–tree and a *morphism* $\phi\colon S \to T$ incorporates a homomorphism $\tilde{\phi}\colon G \to H$, so that if we regard $T$ as a $G$–tree via $\tilde{\phi}$ then $\phi$ is a morphism of $G$–trees. As well as the basic folding operations of [1] it is also necessary to include *vertex morphisms* each of which changes just one vertex label of the corresponding

graph of groups. It is possible to generalize the Bestvina–Feighn result for the case when $\tilde{\phi}$ restricts to an injective homomorphism on point stabilizers of $S$. Under similar restrictions to those specified for a $G$–morphism, $\phi$ is a product of elementary folds, vertex morphisms and an isomorphism. A sequence of such operations is called a *folding sequence*. We can think of each tree in the sequence as the realization of a combinatorial tree. The folding and vertex morphisms correspond to morphisms of the combinatorial trees. If we interpret our folding sequence as a folding sequence of combinatorial trees then we also have to allow subdivision operations. This is because two different combinatorial trees may have isomorphic realizations as $\mathbf{R}$–trees. However if this does happen, then the two trees have isomorphic subdivisions.

Folding sequences are surprisingly useful. They yield theoretical results on decompositions of groups and also provide a way of constructing groups with strange properties.

A $G$–tree $S$ is called reduced if for every edge $e \in ES, G_e = G_{\iota e}$ implies $\iota e, \tau e$ are in the same orbit. Let $S$ be a reduced $G$–tree in which every edge group is finite. Let $\overline{S} = G \backslash S$ and let $(G(-), \overline{S})$ be the corresponding graph of groups. Put

$$\eta(S) = \sum_{e \in E\overline{S}} 1/|G(e)|.$$

Linnell [12] proved that $\eta(S) \leq 2d_G(\omega\mathbf{Q}G) - 1$ where $d_G(\omega\mathbf{Q}G)$ is the minimal number of generators of the augmentation ideal $\omega\mathbf{Q}G$ as a $\mathbf{Q}G$–module. Linnell's argument uses norms in $W^*$–algebras. Using a folding sequence argument we show that $\eta(S) \leq d(G)$, the minimal number of generators of $G$. If all the edge stabilizers of $S$ are trivial, then $\eta(S) = |E\overline{S}|$ and so $|E\overline{S}| \leq d(G)$. This is a weak version of the Grushko–Neumann Theorem (see [4] or [16]). A stronger version of the Grushko–Neumann Theorem is obtained by a closer examination of the folding sequence. Stallings [16] has given a proof of this result using this approach.

Let $G$ be a group. In [8] and [9] I introduced the idea of a $G$–*protree*. A *splitting sequence* of $G$–trees $T_1, T_2, \ldots$ is a sequence such that for each $n$ there is a surjective $G$–map $T_n \to T_{n-1}$ obtained by contracting finitely many orbits of edges. A $G$–protree $P$ arises as the inverse limit of this sequence. As shown in [9], if $ET_n$ is countable for all $n$, then $P$ has a realization as an $\mathbf{R}$–tree, on which $G$ acts by isometries. In this $\mathbf{R}$–tree the set of branch points intersects each segment in a nowhere dense subset. A finitely generated group $G$ is said to be *inaccessible* if there is a splitting sequence of reduced $G$–trees as above, for which all edge groups are finite and the number of $G$–orbits of $VT_n$ (or $ET_n$) tends to infinity. In this case we obtain a $G$–protree $P$ with infinitely many orbits of edges.

We prove in Section 3 that if $G$ is finitely generated and $P$ is a $G$–protree with countably many edges then the realization of $P$ is a direct limit of a folding sequence of simplicial **R**–trees. If the $G_n$–tree $S_n$ is the $n$–th term of the sequence, then there is a surjective homomorphism $\tilde{\rho}_n\colon G_n \to G_{n+1}$ and $G$ is the direct limit of this system of homomorphisms in the category of groups. This description of $G$ gives information as to the subgroup structure of $G$. In particular either $G \cong G_n$ for all sufficiently large $n$ or $G$ must contain a subgroup which is the union of a properly ascending chain of finitely generated subgroups each of which is contained in an edge stabilizer of $P$. It follows that an inaccessible group must contain an infinite locally finite subgroup. If every edge stabilizer of $S_n$ in $G$ is cyclic (not necessarily finite), then $G$ must contain a non-cyclic subgroup that is locally cyclic. It also follows that if $G$ has an infinite splitting sequence then for any integer $k$ there is an integer $n$ such that $G$ contains a nontrivial element which fixes an edge path in $T_n$ of length at least $k$. This is also implied by Sela's results on acylindrical accessibility [14].

Infinite folding sequences were used first by Bestvina and Feighn [2] to give an example of a finitely generated group which had an infinite splitting sequence in which all edge groups are free abelian of rank 2. Subsequently [7], [8], [9] I gave a number of examples of inaccessible groups all of which were constructed (essentially) by means of folding sequences.

Martin and Skora [13] have obtained some results on accessible convergence groups acting on $S^2$. It is not hard to show that an infinite locally finite group cannot act as a convergence group on $S^2$. Hence by Theorem 4.5 a finitely generated convergence group acting on $S^2$ must be accessible. Thus the accessibility condition in the results of Martin and Skora can be removed (or replaced by a finite generation condition). In particular it follows that if $G \subset \mathrm{Hom}(S^2)$ is an orientation preserving convergence group, then there is a simplicial $G$–tree $T$ such that $G\backslash T$ is a finite graph, all edge stabilizers are finite, and if $v \in VT$, then the ordinary set $O(G_v)$ is simply connected.

## 2 Folding

We recall and modify some of the terminology of [6] or [15].

Let $G$ be a group. A *$G$–tree* $T$ is an **R**–tree with $G$ acting on the left by isometries. A $G$–tree is *minimal* if it has no proper $G$–subtree.

Given an **R**–tree $T$ and $x \in T$, define $B_x = \{[x,y]|y \in T - x\}$. Define an equivalence relation on $B_x$ by

$[x, y] \sim [x, z]$ if $[x, y] \cap [x, z] = [x, w]$ for some $w \in T - x$.

A *direction at x* is an element of $B_x / \sim$. There is a bijection between directions at $x$ and the components of $T - x$. A *point of reflection x* of a $G$–tree $T$ is a point with two directions for which there exists $g \in G$ which fixes $x$ and transposes the two directions at $x$. We say that $x \in T$ is an *ordinary point* if there are exactly two directions at $x$ but $x$ is not a point of reflection. A *branch point* is a point $x$ with more than two directions or equivalently for which $T - x$ has more than two components. A *vertex* is a point which is not an ordinary point.

An **R**–tree is *simplicial* if the set of vertices is discrete. For each $x \in T$, let $d(v)$ denote the number of directions at $x$.

A *morphism* from a $G$–tree $S$ to a $G$–tree $T$ is a $G$–map $\phi \colon S \to T$ such that for each segment $[x, y]$ of $S$ there is a segment $[x, w] \subset [x, y]$ such that $\phi|_{[x,w]}$ is an isometry.

Alternatively ([6]) $\phi \colon S \to T$ is a morphism if every segment has a finite subdivision such that $\phi$ restricts to an isometry on each segment of the subdivision.

We generalize the notion of morphism to allow different groups to act on domain and codomain. Thus if $S$ is a $G$–tree and $T$ is an $H$–tree, a *morphism* $\phi \colon S \to T$ is a homomorphism $\tilde{\phi} \colon G \to H$, and a map $\phi \colon S \to T$ such that if we regard $T$ as a $G$–tree via $\tilde{\phi}$ then $\phi$ is a morphism when regarded as a morphism of $G$–trees. Such morphisms are discussed in unpublished work of Skora.

A simplicial **R**–tree $T$ can be regarded as the *realization* of a simplicial complex, which is a (combinatorial) tree. This will also be denoted $T$. Thus $VT$ will correspond to a non-empty closed discrete subset of the **R**–tree containing all branch points and $ET$ will be the set of closures of components of $T - VT$, where VT is such that each element of $ET$ is a closed segment the endpoints of which are elements of $VT$. As a combinatorial tree the vertices of the edge $e$ are denoted $\iota e, \tau e$. When regarded as a protree the edges of a tree are regarded as directed pairs. Usually an edge of a tree is not directed.

Bestvina and Feighn [1] have shown that any morphism of simplicial $G$–trees is a product of subdivisions and folds (which are described as operations on the corresponding combinatorial $G$–trees). Folds are classified according to their effect on the quotient graph. The quotient graph $X = G\backslash T$, together with a labelling by subgroups of $G$ which are the stabilizers of a lift of a maximal subtree $X_0$ of $X$, is known as a graph of groups $(G(-), X)$. See [4] for an account of this theory. The basic folds of Type I, II and III are shown below in Figure 1. These are denoted Type IA, IIA and IIIA in [1]. Bestvina and Feighn
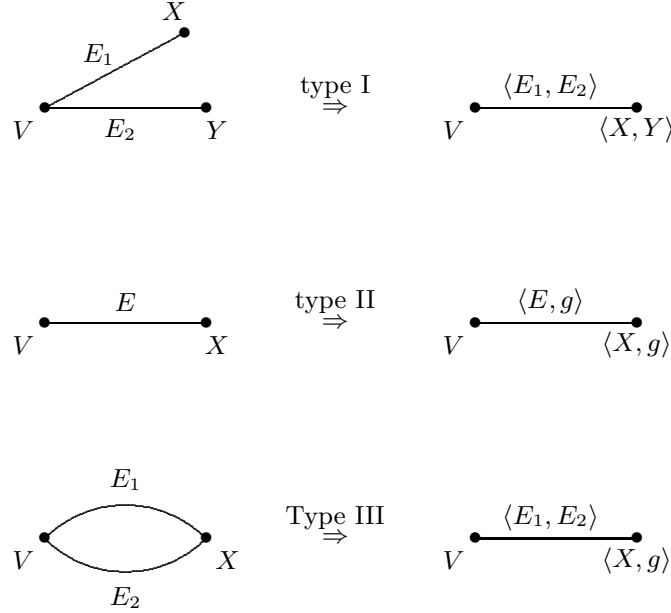
Figure 1

list other basic folds (Type IB,IIB, IIIB and IIIC ). But as they remark, each
of these is equivalent to a combination of Type A folds and subdivisions.

In [9] I introduced vertex morphisms. A *vertex morphism* is a morphism $\theta\colon S \to T$ of simplicial $\mathbf{R}$–trees for which the only change in the corresponding graph
of groups is a change in the label of one of the vertices. Thus if the label $U$
is changed to $V$ then there is a surjective homomorphism $\theta_U\colon U \to V$ which
restricts to the identity map on subgroups which label incident edges. For vertex
morphisms the group $G$ acting on $S$ is different from the group $H$ acting on
$T$. We now generalize the Bestvina–Feighn result to allow different groups to
act on domain and codomain.

**Theorem 2.1**  *Let $S, T$ be simplicial $\mathbf{R}$–trees. Let $G$ act by isometries on $S$
and let $H$ act by isometries on $T$ so that $G\backslash S$ is finite, and all edge stabilizers
of $T$ in $H$ are finitely generated. Also $T$ is a minimal $H$–tree. Let $\phi\colon S \to T$
be a morphism, such that the corresponding homomorphism $\tilde{\phi}\colon G \to H$ is
surjective, and restricts to an injective map on each point stabililizer, then $\phi$
can be written as a product of basic folds and vertex morphisms.*

**Proof**  We adapt the proof of the Proposition in Section 2 of [1].

**Step 0**  We show that if $K$ is a finite simplicial subtree of $S$, then we can factor
$\phi$ as $\gamma\beta$ where $\beta$ is a product of folds and vertex morphisms and $\gamma$ restricted

to $\beta(K)$ is an embedding. Also $\tilde{\gamma}$ is injective on all point stabilizers. If $\phi|K$ is not already an embedding then we can perform a basic fold $\phi_1 \colon S \to S_1$ folding together edges $e_1, e_2$ of $S$ so that $\phi(e_1) = \phi(e_2)$ and $e_1, e_2$ are distinct edges of $X$. The basic fold $\phi_1$ produces at most one new edge group and one new vertex group. The new edge group is a subgroup of an existing vertex group. It follows that $\tilde{\phi}_1$ restricts to an injective homomorphism on the stabilizers of all except one orbit of vertices of $S_1$ and on the stabilizers of all edges. Thus there is a vertex morphism $\nu_1 \colon S_1 \to T_1$ such that $\phi \colon S \to T$ factors $\phi = \phi^{(1)}\nu_1\phi_1$ as a morphism of $\mathbf{R}$–trees (regarding $T$ as an $H$–tree), and also $\tilde{\phi}^{(1)} \colon G_1 \to H$, the homomorphism corresponding to $\phi^{(1)}$, retricts to an injective homomophism on all point stabilizers. Note that $\nu_1\phi_1(K)$ has fewer edges than $K$. We can therefore proceed by induction on the number of edges of $K$.

**Step 1** We now claim that we can factor $\phi$ as $\gamma\beta$ so that $\gamma$ induces a homeomorphism of quotient graphs, $\tilde{\gamma}$ is injective on point stabilizers and $\beta$ is a product of basic folds and vertex morphisms. This follows exactly as in the corresponding argument in [1]. The fact that $T$ is a minimal $H$–tree and $\tilde{\phi}$ is surjective, together imply that the induced morphism $G\backslash S \to H\backslash T$ is a surjective simplicial map. One then uses an induction argument based on the number of edges of $G\backslash S$, using Step 0.

**Step 2** Since edge stabilizers in $T$ are finitely generated, we can use the argument of [1] to show that $\phi$ can be factored $\phi = \gamma\beta$ as in Step 1 and in addition $\tilde{\gamma}$ induces an isomorphism on all edge stabilizers.

**Step 3** It follows as in [1] that the $\gamma$ obtained in Step 2 is an isomorphism. $\square$

We say that in the $G$–tree $S$ an edge $e \in ES$ is *compressible* if $G_{\iota e} = G_e$ and $\iota e$ and $\tau e$ lie in different $G$–orbits. We say that $S$ is *reduced* if it has no compressible edges. For any $G$–finite $G$–tree $S$ there is a reduced $G$–tree $S^*$ for which $VS^*$ is a $G$–retract of $S$: $S^*$ is obtained from $S$ by compressing compressible edges. The retraction is not, in general, uniquely determined. The retraction is determined by a *compressing forest* $F$ defined as follows:

**(1)** $F$ is a subgraph of $G\backslash S = \overline{S}$.

**(2)** The edges of $F$ are oriented (given arrows) so that each vertex $v \in VF$ has at most one arrow pointing away from it.

**(3)** If $e \in EF$ then $G(e) = G(\iota e)$, where the arrow on $e$ points from $\iota e$ to $\tau e$.

**(4)** $F$ is maximal with respect to properties (1), (2) and (3). In particular $VF = V\overline{S}$.

In each component $c$ of $F$ there is exactly one vertex $v_c$ which has no arrow pointing away from it. The retraction $S \to S^*$ corresponding to $F$ induces a retraction $\rho \colon \overline{S} \to \overline{S}^*, \rho(v) = v_c, v \in c$.

It is often convenient to work with reduced trees. We know that it is possible to factorize a morphism of reduced trees as a product of subdivisions, folds and vertex morphisms. Unfortunately subdividing a tree always produces compressible edges. We introduce some modified folding operations which allow us to factorize a morphism of reduced trees so that the intermediate trees obtained are also reduced. These modified folds are shown in Figures 2 ,3 and 4.

Every morphism of $G$–trees is a product of subdivisions and folds of types I, II and III. Let $\phi\colon S \to T$ be such a fold. Given a compressing forest $F$ in $\overline{S}$, we will describe how to construct a compressing forest $F'$ in $\overline{T}$ and describe the corresponding induced morphism $\phi^*\colon S^* \to T^*$. Again these are best described by their effect on the labelled quotient graphs.

Subdivision induces an isomorphism on the corresponding reduced trees, since one enlarges the compressing forest to include half the subdivided edge. Thus a morphism of reduced trees can always be written as a product of isomorphisms and the morphisms $\phi^*\colon S^* \to T^*$ induced by type I, II and III folds. These are discussed in detail below.

We consider the effect of folds on the quotient graph $\overline{S}$ and the quotient reduced graph $\overline{S}^*$. In the subsequent discussion, and in the diagrams of graphs of groups, the group corresponding to a given edge or vertex is denoted with the corresponding capital letter, eg the group corresponding to vertex $v$ is $V$ and the group of $e_1$ is $E_1$. For any vertex $w$, put $\rho(w) = w^*$, which therefore has the group $W^*$. Note that if $W = W^*$ then we can change the arrows on $F$ so that $w$ has no arrows pointing away from it (by reversing all the arrows on the geodesic from $w$ to $w^*$). A change like this has no effect on $\overline{S}^*$.

We now list the different possibilities for the fold $\phi$ and the resulting induced fold $\phi^*$

## Type I

$e_1, e_2 \in F$

We choose the new compressing forest $F'$ to contain all $x \in F, x \neq e_1, e_2$. Also $e_1, e_2$ fold to form the edge $\langle e_1, e_2 \rangle$, which is included in $F'$ with an arrow pointing away from pivot vertex $v$ if and only if one of the edges $e_1, e_2$ has arrow pointing away from $v$. It is easy to check that $F'$ is a compressing forest and $\phi$ induces an isomorphism on $S^*$, since the folding takes place in a part of the tree that is compressed both before and after the fold.
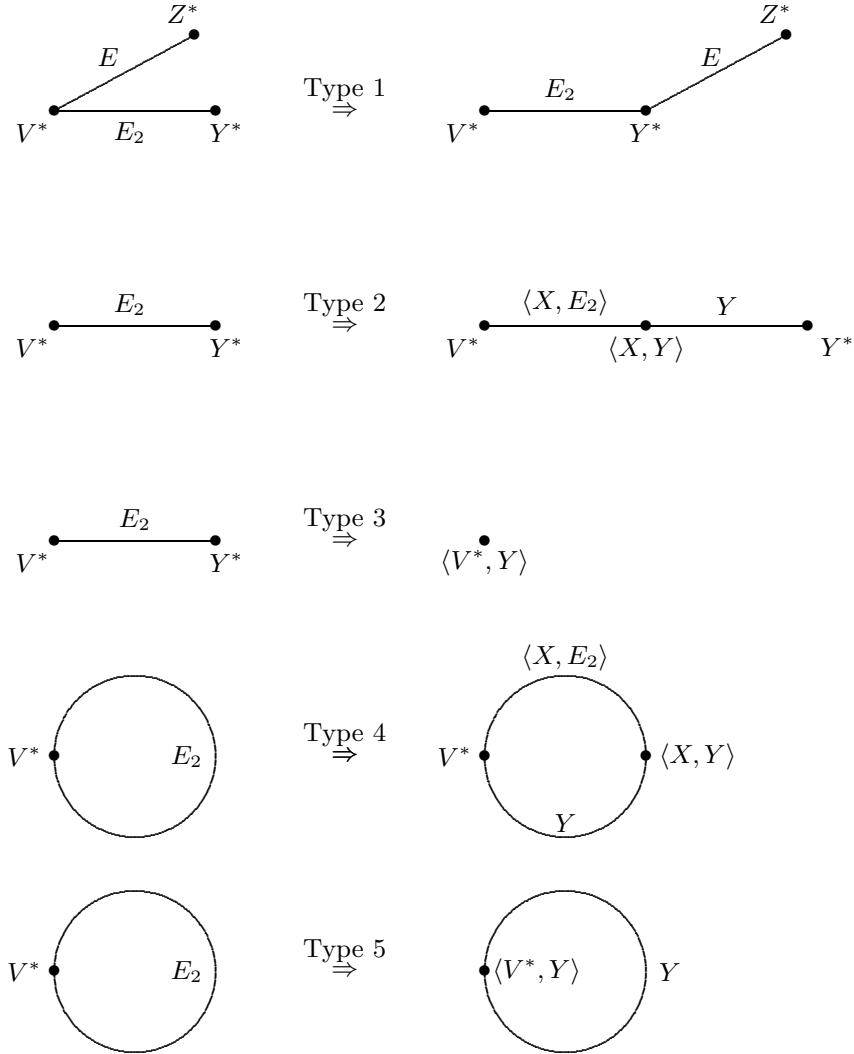
Figure 2

**$e_1 \in F, e_2 \notin F$ and $v, y$ in different components of $F$**

Suppose first that the arrow on $e_1$ goes from $x$ to $v$. Then $X = E_1$. After the fold $F'$ is obtained from $F$ by deleting $e_1$. If $X \leq E_2$, then $\phi^*$ consists of a composite of Type 1 folds for each edge $e$ which has a vertex $w$ in the same component of $F$ as $v$ but for which the arrowed path from $w$ to $v^*$ passes through $x$. It is important to note that in each such Type 1 fold $E \leq E_2$. Assume then that $X \not\leq E_2$. If $\langle X, E_2 \rangle \neq V^*$ then after folding the new compressing forest is obtained by omitting the folded edge and also the edge
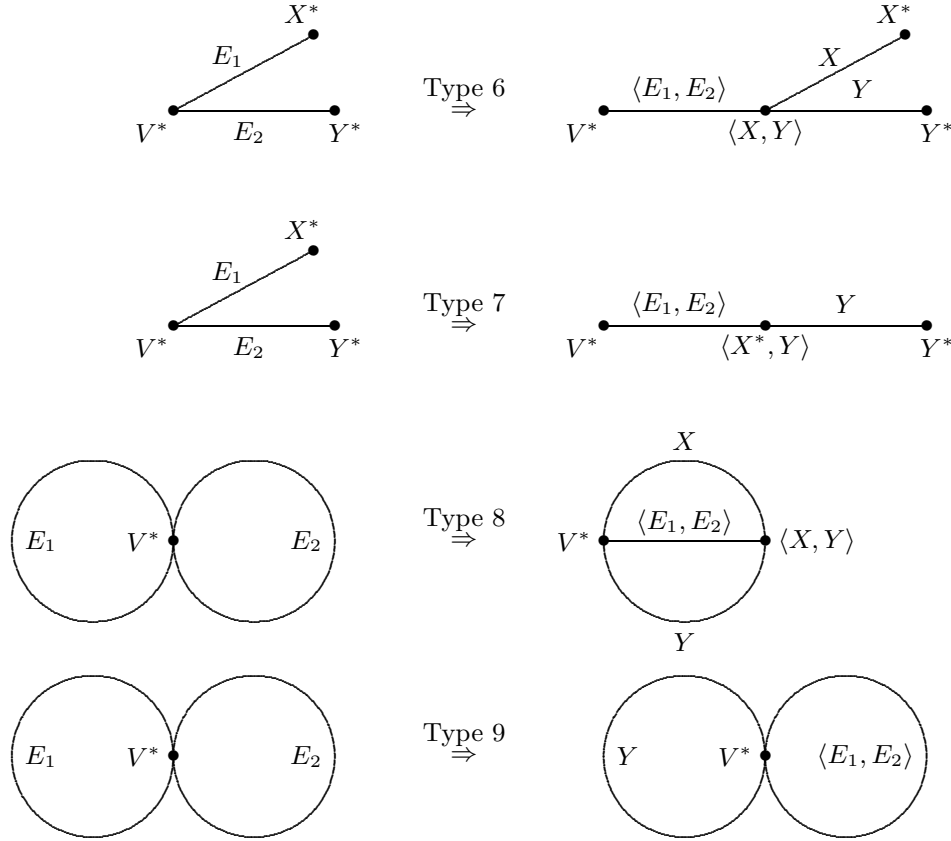
Figure 3

originally pointing away from $y$ if $Y \neq Y^*$. Note that $\langle X, E_2 \rangle \neq \langle X, Y \rangle$, since $\langle X, E_2 \rangle$ is a subgroup of $V^*$ but $Y$ is not contained in $V^*$. Such a fold is called a Type 2 fold. Note that we can assume $E_2 \neq Y$ in a Type 2 fold, since if $E_2 = Y$, then because $v, y$ are in different components of $F$ we could get a bigger compressing forest by adding $e_2$. If $Y = Y^*$, then the induced fold is a combination of Type II folds. Similarly if $\langle X, E_2 \rangle = V^*$ (so that the folded edge must be added to $F$) and $Y \neq Y^*$, then the induced fold is a combination of Type II folds. If $\langle X, E_2 \rangle = V^*$ and $Y = Y^*$ then the induced fold is a Type 3 fold.

If the arrow on $e_1$ goes from $v$ to $x$, then the fold produces a compressible edge which can be included in the the new compressing forest with arrow going from $v$ to $\langle x, y \rangle$. If there are arrows in $F$ pointing away from $x$ and $y$ then these edges must be omitted from the new compressing forest. If $X \neq X^*(= V^*)$ and $Y \neq Y^*$, the effect on $\overline{S}^*$ is a Type 2 fold (with $\langle X, E_2 \rangle = X$). Note that
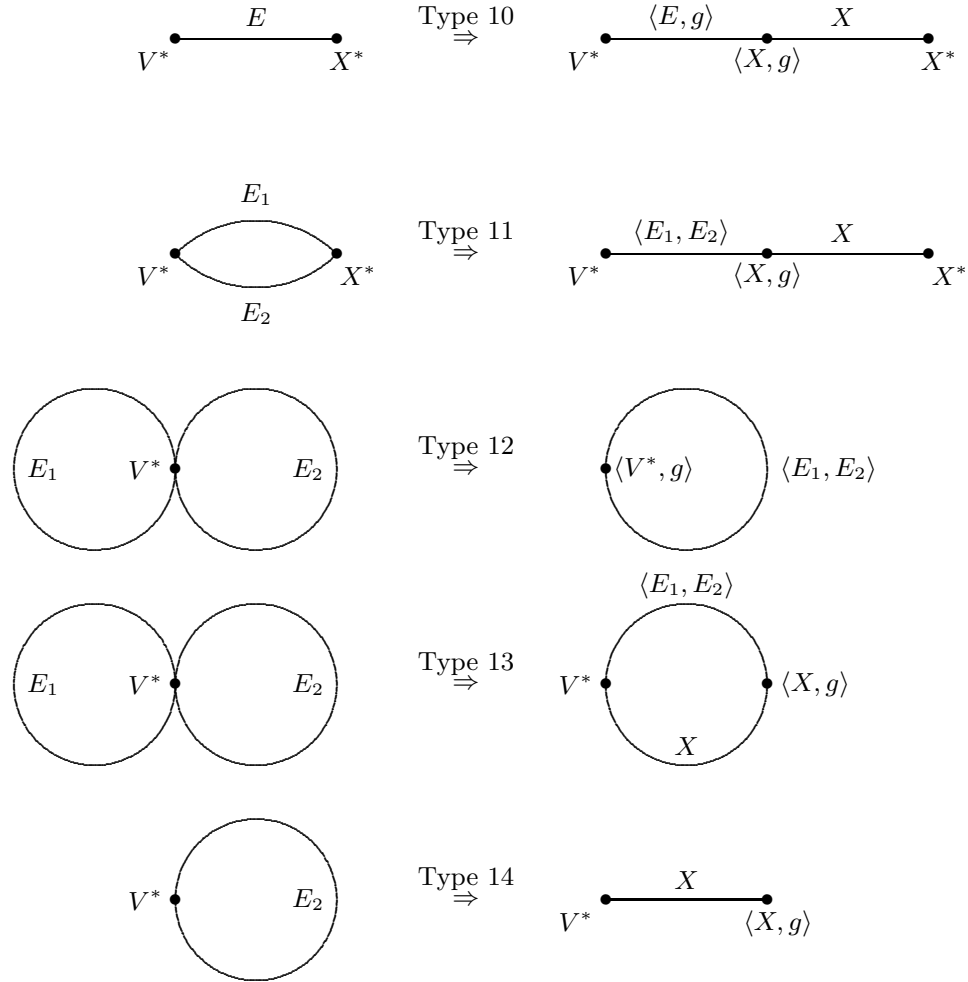
Figure 4

$E_2$ is a proper subgroup of $X$, since otherwise we could add $e_2$ to $F$ and get a larger compressing forest in $S^*$. The induced fold for $X = X^*$ and $Y \neq Y^*$ is a combination of Type II folds (with $y$ as the pivot vertex instead of $v$). The vertex which is initially labelled $V^*$ finishes with label $\langle V^*, Y \rangle$ and the vertex with label $Y^*$ is unchanged. The folded edge becomes a vertex if $X = X^*$ and $Y = Y^*$. Thus we have a Type 3 fold.

**$e_1 \in F, e_2 \notin F$ and $v, y$ in the same component of $F$**

We can assume $E_2 \neq Y$, since if $E_2 = Y$ we could change $F$ so that it included both $e_1$ and $e_2$ which is a case already considered. To see this note that

$v^* = y^*$. If there is no edge of $F$ pointing away from $y$ then $v^* = y$ and $V = Y$ and we can change arrows so that there is an edge in $F$ pointing away from $y$. Now change $F$ so that it includes $e_2$ and omits this edge. Thus we can assume $E_2 \neq Y$. The analysis for this case is now similar to that when $v, y$ are in different components. The induced fold is of Type 4 if $\langle X, E_2 \rangle \neq V^*$ and of Type 5 if $\langle X, E_2 \rangle = V^*$. Note that, since the part of the graph of groups we are concerned with in this case is not a tree, it cannot be assumed that all the edge labels are subgroups of the incident vertex labels. Thus in a Type 4 fold, $Y$ is not assumed to be a subgroup of $V^*$—it is conjugate to a subgroup of $V^*$. There is no analogous case to Type 3.

### $e_1 \notin F, e_2 \notin F, v, x, y$ **in distinct components of** $F$

If either $X = X^*$ or $Y = Y^*$, then we can change the arrows on $F$ so that either $x$ or $y$ has no edges pointing away from it. Thus if $F$ contains edges pointing away from both $x$ and $y$, then we can assume $X \neq X^*$ and $Y \neq Y^*$. In this case we must omit at least one of these edges from $F$ after the fold. If $\langle X, Y \rangle \neq X$ then we must omit the edge of $F$ with initial vertex $x$. Similarly if $\langle X, Y \rangle \neq Y$ then we must omit the edge of $F$ with initial vertex $y$. If $\langle X, Y \rangle = X = Y$ then we need only omit one of the two edges, and we can choose either. First consider the case when both edges are omitted. The fold in this case is a Type 6 fold if $V^* \neq \langle E_1, E_2 \rangle$. Note that $E_1 \neq X$ and $E_2 \neq Y$, since otherwise we could add $e_1$ or $e_2$ to $F$, contradicting its maximality. If $V^* = V = \langle E_1, E_2 \rangle$ then the folded edge is compressible and can be added to $F$. The induced fold in this case is a combination of Type II folds: first operating on the edge $e_1$ by increasing $E_1$ to $X$ and $V^*$ to $\langle V^*, X \rangle$ and then operating on the edge $e_2$ by increasing $E_2$ to $Y$ and $\langle V^*, X \rangle$ to $\langle X, Y \rangle$. For any edge of $\overline{S}$ that is not in $F$ which has a vertex $w$ for which the path from $w$ to $w^*$ passes through $x$ or $y$ it is necessary to carry out a Type 1 fold in the reduced graph. Such an edge, which initially is incident with $x^*$ in $\overline{S}^*$ becomes incident with $\langle x, y \rangle$ in $\overline{T}^*$.

Consider now the case when only one edge is omitted. This happens for example if $X = X^*$ and $Y \neq Y^*$ then the induced fold is of Type 7. If $X = X^*$ and $Y = Y^*$ then the induced fold is just a Type I fold. If $v, x, y$ are in different components of $F$ then both $\langle X, Y \rangle \neq X$ and $\langle X, Y \rangle \neq Y$, since $X \leq Y$ implies $x, y$ are in the same component of $F$. It follows that the edges after the fold cannot be added to $F$.

### $e_1 \notin F, e_2 \notin F, v, x, y$ **not in distinct components of** $F$

This case is similar to the previous case. We can still assume that $E_1 \neq X$ and $E_2 \neq Y$. For if say $E_1 = X$, and $v, x$ are in the same component of $F$, then

either there is an edge in $F$ pointing away from $x$ or $X = V = V^*$ and there is an edge in $F$ pointing away from $v$. We can then change $F$ by removing this edge and replacing it by $e_1$. Such a change induces an isomorphism on the reduced graph. The fold will now involve an edge of $F$ and has been considered previously.

Suppose $v, x, y$ are all in the same component of $F$ so that $V^* = X^* = Y^*$ and $\langle X, Y \rangle \neq V^*, \langle X, Y \rangle \neq X, \langle X, Y \rangle \neq Y$. The induced fold is of Type 8. Again it may be necessary to alter by Type 1 folds the incidence of edges to vertices in $\overline{S}^*$. The similarity with the case when $v, x, y$ are in different components of $F$ is because in both cases $F$ is altered in the same way; by omitting the edges pointing away from the identified vertex $\langle x, y \rangle$. It may now be the case that $\langle X, Y \rangle = X$ say. In this case there would be a compressible edge produced and so we can add an extra edge to $F$ and the induced fold is of Type 9.

## Type II

$e \in F$

In such a fold $V \neq E$ and so the arrow on $e$ must point from $x$ to $v$. We can include the folded edge $\langle e, g \rangle$ in $F'$, with arrow pointing from $\langle x, g \rangle$ to $v$.

$e \notin F, v, x$ **in different components of** $F$

We obtain a Type 10 fold for the case when $X \neq X^*$. Type 1 folds in $\overline{S}^*$ are necessary corresponding to any edge of $\overline{S} - F$ joined to a vertex $w$ for which the path from $w$ to $w^*$ passes through $x$. If $X = X^*$ then the induced fold is just a Type II fold.

$e \notin F, v, x$ **in the same component of** $F$

This is the same as the previous case except that the vertices $v^*$ and $x^*$ are identified before and after the folds. This gives rise to folds of Type 4 and 5.

## Type III

$e_1, e_2 \notin F, v, x$ **in different components.**

We obtain a Type 11 fold when $X \neq X^*$. Again Type 1 folds may be neccessary corresponding to any edge of $\overline{S} - F$ joined to a vertex $w$ for which the path from $w$ to $w^*$ passes through $x$. If $X = X^*$ then the induced fold is just a Type III fold.

$e_1, e_2 \notin F, v, x$ **in the same component of** $F$

This produces a Type 12 fold if $X = X^*(= V^*)$, and a Type 13 fold if $X \neq X^*$.

$e_1 \in F, e_2 \notin F$

In this case, since $e_2$ has both its vertices in the same component of $F$ it may be the case that $E_2 = X$. We obtain a Type 14 fold.

We see then that the induced folds in reduced trees may just be a Type I, II or III fold, but it may be of a type which creates a new vertex. For example a Type 6 fold creates a new vertex.

Theorem 2.1 can now be adapted for morphisms between reduced trees.

**Theorem 2.2** *Let $S, T$ be simplicial reduced $\mathbf{R}$–trees. Let $G$ act by isometries on $S$ and let $H$ act by isometries on $T$ so that $G \backslash S$ is finite, and all edge stabilizers of $T$ in $H$ are finitely generated. Also $T$ is a minimal $H$–tree. Let $\phi \colon S \to T$ be a morphism, such that the corresponding homomorphism $\tilde{\phi} \colon G \to H$ is surjective, and restricts to an injective map on each point stabililizer, then $\phi$ can be written as a product of folds of Type I, II and III or of Types 1 – 14 and vertex morphisms and all the intermediate trees are reduced.*

This result enables us to deduce certain bounds on the complexity of decompositions of finitely generated groups.

Let $S$ be a $G$–tree with finite edge stabilizers. Define

$$\eta(S) = \sum_{e \in E\overline{S}^*} 1/|G(e)|.$$

**Theorem 2.3** *Let $G$ be a finitely generated group for which $d(G)$ is the minimal number of generators, then $\eta(S) \leq d(G)$.*

**Proof** Let $W$ be a free group of rank $d(G)$ and let $X$ be the $W$–tree with one orbit of vertices on which $W$ acts freely, and for which $\eta(\overline{X}^*) = d(G)$. We regard both $X$ and $S$ as simplicial $\mathbf{R}$–trees. A surjective homomorphism $\tilde{\alpha} \colon W \to G$ induces a morphism $\alpha \colon X \to S$. By Theorem 2.1 $\alpha$ is a product of basic folds and vertex morphisms. We consider the induced folds on the reduced trees. One can check without too much difficulty that $\eta(S)$ does not increase for each of the induced folds described above. For example, for a fold of Type 6

$$\eta(S) - \eta(T) = \frac{1}{|E_1|} + \frac{1}{|E_2|} - \frac{1}{|\langle E_1, E_2 \rangle|} - \frac{1}{|X|} - \frac{1}{|Y|}.$$

We can assume $|E_1| \leq |E_2|$. Also we know that $E_1 < X$ and $E_2 < Y$. Thus $\frac{1}{|X|} \leq \frac{1}{2|E_1|}$ and $\frac{1}{|Y|} \leq \frac{1}{2|E_2|} \leq \frac{1}{2|E_1|}$, so that $\frac{1}{|X|} + \frac{1}{|Y|} \leq \frac{1}{|E_1|}$. Also

$\frac{1}{|\langle E_1, E_2 \rangle|} \leq \frac{1}{|E_2|}$. It is clear in this case that $\eta(S) - \eta(T) \geq 0$. Similar arguments show that $\eta(S)$ does not increase in each of the other cases. A vertex morphism will leave edge groups unchanged and cannot increase $\eta(S)$. The theorem is proved. $\square$

Let us consider the case when $G$ is a finitely generated group and $S$ is a $G$–tree with trivial edge stabilizers. In this case $\eta(S) = |E\overline{S}^*|)$, and so we see that the number of edge orbits in a minimal reduced $G$–tree is bounded by $d(G)$. In fact we obtain stronger versions of the Grushko–Neumann Theorem by examining the folding sequence in this case. Thus we obtain the following theorem, first obtained in [4, I, 10.3].

**Theorem 2.4** *Let $S$ be a $G$–tree and let $T$ be a reduced minimal $H$–tree for which $G$ acts freely on $ES$ and $H$ acts freely on $ET$. Also suppose $H$ is finitely generated. Let $\alpha\colon S \to T$ be a morphism. If $\tilde{\alpha}\colon G \to H$ is surjective then there is a $G$–tree $S'$ and a morphism $\alpha'\colon S' \to T$ that induces an isomorphism $G\backslash S' \to H\backslash T$ and $\tilde{\alpha}'$ induces a surjective homomorphism $G_v \to H_{\alpha'(v)}$ for each vertex $v \in VS'$.*

**Proof** We can carry out vertex morphisms on $S$ and replace each vertex stabilizer by its image under $\tilde{\alpha}$. We will then have a $\hat{G}$–tree $\hat{S}$ for which there is a morphism $\hat{\phi}\colon \hat{S} \to T$ for which the corresponding homomorphism $\hat{G} \to H$ is injective on all point stabilizers. By Theorem 2.1 $\hat{\phi}$ is a product of basic folds, subdivisions and vertex morphisms. We consider the induced operations on the corresponding reduced trees. Since all edge groups are trivial, the only possible induced folds that can occur are Type I, III, 1, 3 and 5 (with $E_2 = X = \{1\}$). If we carry out the same sequence of induced folds on $S^*$ (leaving out all the vertex morphisms), we will obtain the $G$–tree $T'$ with the required properties. $\square$

# 3  Folding sequences

A folding sequence $(T_n)$, is a sequence of combinatorial trees $T_n$, satisfying the following conditions:

**(a)**  $T_n$ is a minimal $G_n$–tree, where $G_n$ is finitely generated.

**(b)**  $T_{n+1}$ can be obtained from $T_n$ either by subdivision, or by a I, II or III fold followed by a vertex morphism.

It is often the case that corresponding to a folding sequence $(T_n)$ is a *folding sequence of simplicial* **R***–trees*, in which we replace each tree by a realization

and the folding operations induce morphisms of **R**–trees. In this case we will risk confusion by using $T_n$ to denote both the tree and its realization as an **R**–tree. There are examples of folding seqences which cannot be realized in the above way. For example if for each $n$, $G_{2n-1}\backslash T_{2n-1}$ is a tree with two edges $e_{2n-1}, f_{2n-1}$, and $T_{2n}$ is obtained from $T_{2n-1}$ by subdividing $e_{2n-1}$ into two edges $e_{2n}$ and $e_{2n+1}$. Then $T_{2n+1}$ is obtained from $T_{2n}$ by a Type I fold, in which $e_{2n}$ and $f_{2n-1}$ are folded together to form $f_{2n+1}$. We call such a folding sequence *reducible*. Thus a folding sequence is reducible if it satisfies the following condition:

There exists $n$, such that for each $m \geq n$ there is a proper subset $E_m \subset ET_m$ which is invariant under $G_m$ and such that if the folding operation involves an edge of $E_m$ then the resulting edges are in $E_{m+1}$.

Thus if the folding operation is subdivision of an edge of $E_m$, then the resulting edges are all in $E_{m+1}$; and if the operation is a Type I fold in which one of the edges is in $E_m$, then the resulting edge is in $E_{m+1}$. In the the above example the folding sequence is reducible since the sets $E_{2m} = E_{2m-1} = \{f_{2m-1}\}$, satisfy the above condition. A folding sequence is *irreducible* if it is not reducible.

**Theorem 3.1** *Let $(T_n)$ be an irreducible folding sequence of combinatorial trees. The sequence can be realized as a folding sequence of morphisms of simplicial **R**–trees in which group actions are by isometries.*

**Proof** For each $n$ it is possible to realize the finite folding sequence $T_1, T_2, \ldots,$ $T_n$ as a folding sequence of morphisms of simplicial **R**–trees in which the group actions are by isometries. To produce such a realization one has to assign a common length to the edges in each orbit of edges in such a way that the lengths are compatible with subdivision and so that Type I and Type III folds take place between edges of equal length. To achieve such a realization assign lengths to the edges of $T_n$ and work backwards, noting that the lengths of edges of $T_i$ are determined by the lengths of edges of $T_{i+1}$. For each $n = 1, 2, \ldots,$ let $z_n = (\xi_{n1}, \xi_{n2}, \xi_{n3}, \ldots, \xi_{nk})$ be the length of the edges $e_1, e_2, \ldots e_k$ of $G_1\backslash T_1$ in such a solution. We may assume that for each $n, |z_n| = \sum_{i=1}^k \xi_{ni} = 1$. By compactness for the standard $n-1$–simplex $|\sigma_{n-1}|$, the sequence $z_n$ has a convergent subsequence. Let $w_1 = (\xi_1, \xi_2, \ldots \xi_k)$ be the limit point of a convergent subsequence. Note that some of values $\xi_i$ may be zero, but not all. We now repeat the above process. For each term of the convergent subsequence for $w_1$, we can find a vector corresponding to a solution for the edges of $G_2\backslash T_2$. The lengths of these vectors is bounded, since $|w_1| = 1$. Again by compactness there is a convergent subsequence converging to $w_2$ and assigning the coefficients of $w_2$ to $G_2\backslash T_2$ will be consistent with assigning the coefficients of $w_1$ to

the lengths of the edges of $G_1 \backslash T_1$. Note that if an edge has been assigned zero length then when subdivided the parts have zero length and it can be part of a Type I fold with another edge of zero length. Again repeating this process we can eventually assign lengths to all the edges of $G_n \backslash T_n$ for every $n$ which are consistent with the folding process. If all these lengths are non-zero then we have realized the folding sequence as a folding sequence of simplicial **R**–trees. If some of the edges have zero length assigned to them, then it is easy to see that the folding sequence is reducible. Thus we take $E_m \subset ET_m$ to be the set of edges assigned zero length. ◻

It is easy to construct the *limit* of such a folding sequence of **R**–trees. Let $\theta_n = \rho_n \rho_{n-1} \ldots \rho_1 \colon T_1 \to T_{n+1}$. Let $d_n$ be the **R**–tree metric in $T_n$. We define a pseudometric $d$ in $T_1$ by $d(x,y) = \lim_{n \to \infty} (d_n(\theta_n(x)), d_n(\theta_n(y)))$. We put $T = T_1 / \sim$, where $x \sim y$ if $d(x,y) = 0$. Clearly $d$ induces a metric on $T$ and this metric space is called the limit of the folding sequence.

I am grateful to Brian Bowditch for supplying the proof of the following theorem.

**Theorem 3.2** *The limit $T$ of the folding sequence $T_n$ is an **R**–tree.*

**Proof** Let $(S, d)$ be a metric space. In the terminology of [3], $d$ is a path metric if given any two points $X, Y \in S$ and $\epsilon > 0$ there is a rectifiable path joining $X$ and $Y$ of length at most $d(X, Y) + \epsilon$. Each $(T_n, d_n)$ satisfies the stronger condition that any two points $X, Y \in T_n$ are joined by a path of length $d(X, Y)$. Since for any $x, y \in T_1, (d_n(\theta_n(x)), d_n(\theta_n(y)))$ is a decreasing sequence, it follows easily that $d$ as defined above is a path metric on $T$. It now follows from [3] Proposition 3.4.2 that $T$ is an **R**–tree if given any four points $X, Y, Z, W$ they can be partitioned into two sets of two elements, without loss of generality, $\{\{X, y\}, \{Z, W\}\}$, so that

$$d(X, Y) + d(Z, W) \le d(X, Z) + d(Y, W) = d(Y, Z) + d(X, W).$$

Since this condition is satisfied in each $T_n$, it must also be satisfied in $T$. Thus $T$ is an **R**–tree. ◻

If $G$ is the direct limit in the category of groups of the sequence of homomorphisms $\rho_n \colon G_n \to G_{n+1}$ then there is an action of $G$ on $T$ via isometries. Suppose in addition the folding sequence satisfies the following condition

**(c)** Two edges of $T_n$ cannot be folded together if they arose as subdivided parts of the same edge of $T_m$ for some $m < n$.

In this case the natural map $\phi_n \colon T_n \to T$ restricts to an isometry on each edge of $T_n$ and it is therefore a morphism of **R**–trees. It is easy to check that $T$

is the direct limit of the sequence of folding morphisms in the category $\mathcal{T}$ of **R**–trees and morphisms.

As noted above, it is best to describe folding operations in terms of their effect on the quotient graphs $G_n \backslash T_n$. Note that (c) applies to $T_n$ and not to $G_n \backslash T_n$. Thus it is possible for the $n$-th fold in the folding sequence to fold together edges that arose as subdivided edges of $G_m \backslash T_m$ for some $m < n$. An example of this is given in [8]. What happens is that, in $T_n$, the edges folded together occur as subdivided parts of different edges in the same $G_m$–orbit in $T_m$.

Let $G$ be a finitely generated group. Suppose we have an infinite folding sequence with limit $T$ and suppose that $\tilde{\phi}_n \colon G_n \to G$ is not an isomorphism for any $n$. This means that the folding sequence must have infinitely many vertex morphisms. There is then an induced folding sequence of reduced trees. We examine the induced folds listed above. For induced folds of type I, III and 3 there is a decrease in the number of orbits of edges. For a fold of type 12, 13 or 14 there is a decrease in the rank of $H_1(\overline{S}^*)$ and for a fold of type 1 there is no change in vertex groups. Thus the sequence must contain infinitely many induced folds of types other than I, III, 1, 3, 12, 13 or 14. However each such induced fold, which is not an isomorphism, produces a new edge group that properly contains one of the old edge groups. In the situation when the maps $\phi_n \colon T_n \to T$ are morphisms of **R**–trees, for example if condition (c) is satisfied, each edge stabilizer of $T_n$ fixes an arc of $T$. Since each $T_n$ has finitely many orbits of edges, using a graph theoretic argument (König's Lemma) it is possible to find a sequence of edge stabilizers from a subsequence of the $T_n$'s for which the inclusions are proper. It follows that $G$ contains a subgroup $H$ that is not finitely generated but every finitely generated subgroup of $H$ fixes an arc of $T$. Thus we have the following result.

**Theorem 3.3** *Let the $G$–tree $T$ be the direct limit in $\mathcal{T}$ of the folding sequence $T_n$ of simplicial trees, where $T$ is a $G_n$–tree. Then either there exists $m$ such that $G = G_n$ for all $n \geq m$ or $G$ contains a subgroup $H$ that is not finitely generated but every finitely generated subgroup of $H$ fixes an arc of $T$.*

In [8] I introduced the concept of a $G$–protree. Protrees arise naturally in studying inaccessible groups. Let $G$ be a finitely generated group. Let $\mathcal{B}(G)$ denote the Boolean ring consisting of all subsets $a \subset G$ of almost invariant sets. Thus $a \in \mathcal{B}(G)$ if and only if the sets $a$ and $ag$ are almost equal for every $g \in G$. In [4] it is shown that there is a *nested $G$–set* $E$ which generates $\mathcal{B}(G)$ as a Boolean ring. The group $G$ is accessible if and only if $E$ can be chosen to be $G$–finite, in which case $E$ can be regarded as the edge set of a

simplicial $G$–tree. If $G$ is inaccessible then $E$ is not $G$–finite. In this case $E$ is a combinatorial object called a nice $G$–protree, which has a realization (also called a $G$–protree) as an **R**–tree in which the set of branch points intersects each segment in a nowhere dense subset.

If $G$ is finitely generated, then any $G$–tree $T$ is a strong limit of a sequence $T_n$ of **R**–trees, where $T_n$ is a $G_n$–tree and the action is geometric, ie it arises from a foliation on a finite 2–complex. See [11] for a precise definition and a proof of the above statement. However in a geometric action an orbit which is nowhere dense must be discrete (see [11]). Thus if $G$ is finitely generated and $T$ is a $G$–protree, then $T$ is a strong limit of a folding sequence of simplicial trees. This gives the following result.

**Theorem 3.4** *Let $G$ be a finitely generated group and let $P$ be a nice $G$–protree. Then either*

(i) *there is a reduced $G$–tree $T$ such that for every $v \in VT, G_v$ is finitely generated and fixes a vertex of $P$ and for every $e \in ET, G_e$ is finitely generated and fixes an edge of $P$,*

*or*

(ii) *the group $G$ contains a subgroup $H$ that is not finitely generated but every finitely generated subgroup of $H$ fixes an edge of $P$.*

Note that if $G$ is finitely presented then $\tilde{\phi}_n$ must be an isomorphism for $n$ large and so (i) must hold. This can be used to give a proof that finitely presented groups are accessible. This was first proved in [5]. We have seen that if $G$ is finitely generated then we can construct a $G$–protree $P$ corresponding to a nested set of generators of $\mathcal{B}(G)$. There is then a folding sequence which has limit $P$. If the situation (i) of Theorem 3.4 prevails then for each $v \in VT, G_v$ will have at most one end and so $G$ will be accessible. Thus if $G$ is inaccessible then condition (ii) must be satisfied giving the following result.

**Theorem 3.5** *Let $G$ be a finitely generated inaccessible group. Then $G$ contains an infinite locally finite subgroup.*

**Proof** This follows immediately from Theorem 3.4. $\square$

**Corollary 3.6** *Let $G$ be a finitely generated discrete convergence group acting on $S^2$. Then $G$ is accessible.*

**Proof** By Theorem 3.5 it suffices to show that a locally finite discrete convergence group must be finite. Suppose that $H$ is an infinite locally finite discrete convergence group acting on $S^2$. By [10] Theorem 5.11, $L(H)$ (the set of limit points of $H$) consists of exactly one point $x_0$, which is fixed by $H$. A finite group of homeomorphisms with a fixed point is conjugate in $Hom(S^2)$ to a cyclic or dihedral group acting linearly on $S^2$. An increasing chain of such groups would have to have two fixed points, contradicting the statement above that there is a unique fixed point.

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*Faculty of Math.Studies*
*University of Southampton,*
*Southampton, SO9 5NH, UK*

Email:  `mjd@maths.soton.ac.uk`